Characterization of One-Dimensional Luttinger Liquids in Terms of Fractional Exclusion Statistics

Yong-Shi Wu [1] and Yue Yu [2] and Huan-Xiong Yang[3,2]
1. Department of Physics, University of Utah, Salt Lake City, UT 84112, U.S.A.
2. Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100080, P. R. China
3. Department of Physics, Zhejiang University, Hangzhou, 310027, P. R. China

We develop a bosonization approach to study the low temperature properties of one-dimensional gas of particles obeying fractional exclusion statistics (FES). It is shown that such ideal gas reproduces the low-energy excitations and asymptotic exponents of a one-component Luttinger liquid (with no internal degrees of freedom). The bosonized effective theory at low energy (or temperature) is identified to a $c = 1$ conformal field theory (CFT) with compactified radius determined by the statistics parameter λ. Moreover, this CFT can be put into a form of the harmonic fluid description for Luttinger liquids, with the Haldane controlling parameter identified with the statistics parameter (of quasi-particle excitations). Thus we propose to use the latter to characterize the fixed points of 1-d Luttinger liquids. Such a characterization is further shown to be valid for generalized ideal gas of particles with mutual statistics in momentum space and for non-ideal gas with Luttinger-type interactions: In either case, the low temperature behavior is controlled by an effective statistics varying in a fixed-point line.

I. INTRODUCTION

It is well-known that the Landau theory of Fermi liquids fails to describe most of one-dimensional (1-d) interacting many-body systems. To provide a substitute, Haldane proposed, years ago, the concept of the Luttinger liquid [1], defined by a set of low-lying excitations and critical exponents of the asymptotic correlation functions. Like Fermi liquids, there is a (pseudo-)Fermi surface for the quasiparticle-like excitations in Luttinger liquids, so that the classification of low-lying excitations is similar to that in Fermi liquids. However, the exponents of the asymptotic correlation functions (at low temperature) are distinct from those for Fermi liquid theory. For one-component systems (without internal degrees of freedom), the low-energy or low-temperature behavior of a Luttinger liquid is controlled by a single parameter, the Haldane controlling parameter. It controls not only all exponents, but also the velocity ratios between different types of elementary excitations. The Fermi liquid theory is a special case of the Luttinger liquids with the Haldane parameter $\lambda = 1$.

In recent years, the failure of Landau's theory of Fermi liquids to describe several newly discovered strongly correlated electron systems have revived the interests in the theory of Luttinger liquids. Among other questions, compared with Fermi liquids, one would like very much to know the answer to the following questions:

- What is the physical meaning of Haldane's controlling parameter? Or more precisely, how to use physical properties of low-lying excitations to characterize the concept of Luttinger liquids?
- Does Haldane's theory of Luttinger liquids possess universality in one dimension, just like Landau's theory of Fermi liquids in three dimensions? Or equivalently, in terms of modern language of renormalization group, does the Luttinger liquids describe the infrared (or low-energy) fixed points in 1-d systems?
- In what directions could one expect to go for generalizing the concept of Luttinger liquids to higher than one dimensions?

In short, a characterization of 1-d Luttinger liquids, other than using a bunch of excitations and exponents, is in demand for gaining more insights and looking for possible generalization.

To achieve this, let us recall what motivated Landau's concept of Fermi liquids, which is known to describe an infrared fixed point (or a universality class) of interacting electron systems. The basic idea behind it is based on the following *organizing principle* for interacting many-body systems: *At low temperature, the low-lying excited states of an interacting many-body system above a stable ground state can be viewed as consisting of weakly coupled elementary excitations.* Here "weakly coupled" only means that the total energy can be written as a sum of single-particle (dressed) energies, while the dispersion of the dressed energy may well depend on the total particle number, a signal of remnant interactions between the quasiparticles. According to Landau, the ground state and the low-lying excited states of a Fermi liquid are approximately, to a good accuracy at sufficiently low temperature, described by those of an ideal Fermi gas with dressed energy for the quasiparticles.

We note the significant role played by the ideal Fermi gas distribution (with dressed energy) in this description of Fermi liquids. Actually it is the ideal Fermi gas that gives a characterization to the Fermi liquid fixed point, and a meaning to the universality of the concept of Fermi liquids. This inspires us to try to give a characterization of the 1-d Luttinger liquids along a similar line of thoughts, namely using a properly generalized concept of *exclusion statistics*, of which a special case is the usual Fermi statistics. Because the concept of quantum statistics in statistical mechanics is independent of the dimensionality of a system, a characterization of 1-d infrared fixed points using statistics, if successful, would shed light on how to generalize to higher dimensional non-Fermi liquids.

Fortunately, a generalization as such has been available recently, under the name of fractional exclusion statistics (FES). It is based on a new combinatoric rule for the many-body state counting [2,3], which is essentially an abstraction and generalization of Yang-Yang's state counting [4,5] in 1-d soluble many-body models. FES has been shown to be applicable to elementary excitations in a number of exactly solvable models for strongly correlated systems [2,3,5–9], anyons in the lowest Landau level [10,3], and quasiparticle excitations in the fractional quantum Hall effect [2,3,7,11]. The thermodynamics of the so-called generalized ideal gas (GIG) associated with FES have been studied [3] in a general framework.

Inspired by these results, the thoughts along the lines indicated in the above paragraphs have led two of present authors [12] to propose that at least *for some strongly correlated systems or non-Fermi liquids, their low-energy or low-temperature fixed point may be described by a GIG associated with FES*, similar to the way that of the Fermi liquid fixed point by the ideal Fermi gas [13]. As a testimony to this proposition, a sketchy proof was given in that short letter [12] that the low-T critical properties of the 1-d Luttinger liquids are exactly reproduced by those of 1D ideal excluson gas(IEG), if one identifies the Haldane parameter of the former with the statistics parameter λ of the latter. (We call the particles obeying the FES without mutual statistics *exclusons*). Threfore *IEG can be used to describe the fixed points of the Luttinger liquids.* In this paper, we will present our results obtained in [12] in details, much of which was not published before.

A main tool we use in this study is bosonization of the 1-d excluson systems at low T, *à la* Tomonaga [14] and Mattis and Lieb [15]. To bosonize an IEG system is a little bit tricky, because at low temperature the linearized dispersion of dressed energy versus pseudo-momentum has different slope outside and inside the pseudo-Fermi sea: There is 'refraction' at both pseudo-Fermi points. In spite of this, we still manage to construct well-defined density fluctuation operators that obey the $U(1)$ current algebras and physically describe free phonons. Then, the Tomonaga-Mattis-Lieb bosonization applies, resulting a bosonized effective field theory, in agreement with Haldane's harmonic fluid description of the Luttinger liquid [16]. Then the asymptotic correlation functions and their exponents can be systematically calculated. In this way, the critical properties of IEG reproduce those of the Luttinger liquids.

An important consequence of our bosonization is that the low energy behavior of IEG is controlled by an *orbifold* conformal field theory (CFT) with central charge $c = 1$ and compactified radius [19] $R = \sqrt{1/\lambda}$. This variant of $c = 1$ CFT is *not* the ordinary $c = 1$ CFT compactified on a circle S^1, rather it is compactified on an orbifold S^1/Z_2, which is topologically an interval [19]. The differences arise due to different selection rules for vertex operators, that constrain quantum numbers of possible quasiparticle excitations in the system. In the usual literature this difference quite often is overlooked. Only within the orbifold CFT the IEG with statistics parameter $\lambda = 1$ recovers ideal Fermi gas, as it should be. Also the two classes of $c = 1$ CFT's have different duality relation; only the one in orbifold CFT reproduces the known particle-hole duality in IEG, $\lambda \leftrightarrow 1/\lambda$, as given in [5,6]. (For the details and more elaboration, see below. We note [9] that a similar situation happens for the Calogero-Sutherland (C-S) model [17,18]: The low-energy effective field theory for the bosonic and fermionic C-S models belongs to, respectively, the above-mentioned two classes of $c = 1$ CFT.)

The fact that the low-T behavior of IEG is controlled by a *conformally invariant* theory is significant, implying that indeed IEG provides a characterization of infrared fixed points, having the conformal invariance as required by renormalization group. We have also studied the effects of mutual statistics between different pseudomomenta and of the Luttinger-type (density-density) interactions among exclusons. In either case, the low-T behavior is controlled by an effective statistics λ_{eff} for excitations near the Fermi points, the same way as λ in the case of IEG. In one dimension both the momentum-independent part of interactions and change in chemical potential μ are *relevant* perturbations [13,25], leading to a continuous shift in the fixed-point line parameterized by λ. All these will be explained in details in the present paper.

To make this paper self-contained, we devote the next two sections, Sec. II and Sec. III, to reviewing the Luttinger liquid theory and the GIG associated with FES, respectively. In Sec. IV, we discuss the low-energy behavior of the IEG system and achieve its bosonization. In Sec. V, the generalization to the GIG with mutual statistics as well as the non-ideal gas with FES are provided. The last section is dedicated to conclusions and discussions.

II. LUTTINGER LIQUID

The Luttinger liquid, which describes a very large class of one-dimensional interacting many-body systems, is introduced because of the infrared divergence of certain vertices in the Fermi liquid description of the 1-d systems. Some pioneering works have been done in the Luttinger

UU IR Author Manuscript

model before the Luttinger liquid concept [26,15]. The model has been exactly solved by using the bosonization technique [15]. Haldane [1] re-solved the model with the following important observations:

(i) Besides a linearized spectrum of non-zero mode excitations, i.e., the density fluctuations (sound waves), there are two kinds of zero mode excitations, single-particle excitations by adding extra particles to the system and persistent currents by making Galileo boosts.

(ii) There is a fundamental relation among the velocities of these three types of excitations

$$v_s = \sqrt{v_N\, v_J}, \tag{2.1}$$

where v_s is the sound velocity, v_J the current velocity and v_N a velocity related to the change in particle number. The velocity ratios define a controlling parameter, $e^{-2\varphi}$, by

$$v_N = v_s e^{-2\varphi}, \qquad v_J = v_s e^{2\varphi}. \tag{2.2}$$

(iii) The above defined controlling parameter measures the essential renormalized coupling constant, and is the unique parameter that determines the exponents of power-law decay in the zero-temperature correlation functions.

Based on these observations, Haldane defined the Luttinger liquids as 1-d systems that have similar behavior (i)-(iii) at low temperature just like the Luttinger model. In this way the Luttinger liquids are characterized through their excitations and the exponents of the asymptotic correlation functions.

To be more precise, recall that the Luttinger model describes a one-dimensional interacting fermion system with the Hamiltonian

$$H = \int dx |\nabla \psi|^2 + \frac{1}{2} \int \int dx dy V(x-y) \rho(x) \rho(y). \tag{2.3}$$

In the low energy limit, the Hamiltonian (2.3) can be bosonized as

$$H = v_s \sum_q |q| b_q^\dagger b_q + \frac{1}{2} (\pi/L)(v_N\, M^2 + v_J\, J^2), \tag{2.4}$$

where b_q are the standard boson annihilation operators, and M and J the operators corresponding to adding extra particle and boosting persistent currents, whose eigenvalues obey the following selection rule,

$$(-1)^J = (-1)^M. \tag{2.5}$$

The total momentum of the system also has a bosonized form

$$P = [k_F + (\pi/)M]J + \sum_q q b_q^\dagger b_q, \tag{2.6}$$

with k_F being the Fermi momentum.

Eqs.(2.1,2.2,2.4-2.6) turned to be universally valid for the description of the low-energy properties of gapless interacting one-dimensional spinless fermion systems even for those not exactly soluble with a conserved current J. This universality class is named as the Luttinger liquid by Haldane [1]. The Luttinger liquid has a model-independent representation, namely the harmonic fluid description [16], which is convenient for calculating the correlation functions. The results of the harmonic fluid representation are listed in the Appendix, for later use to be compared with our bosonization theory of the IEG.

The Haldane theory of Luttinger liquids is based on the significant observation that the low-T behavior of the Luttinger model is universal. Naturally arises the question: Why is it so? In this paper we intend to answer this question by pointing out a profound coincidence of the low-T behavior of the Luttinger model and that of ideal excluson gas (IEG), i.e., ideal gas of particles obeying fractional exclusion statistics: The universality of the former is due to that of the latter.

III. GENERALIZED IDEAL GAS

In quantum mechanics, there are two ways to define the statistics of particles. One is in terms of the symmetry of the many-body wave function under particle exchange. The other is based on the state counting. Here we are interested in the latter definition. As is well-known, bosons and fermions have different countings for many-body states, or different statistical weights W: The number of quantum states of N particles occupying a group of G states is, for bosons and fermions respectively, given by

$$W_b = \frac{(G+N-1)!}{N!\,(G-1)!}\,, \quad \text{or} \quad W_f = \frac{G!}{N!\,(G-N)!}\,. \tag{3.1}$$

A simple interpolation between bosons and fermions is given by [2,3]

$$W = \frac{[G+(N-1)(1-\lambda)]!}{N!\,[G-\lambda N-(1-\lambda)]!}\,, \tag{3.2}$$

with $\lambda = 0$ corresponding to bosons and $\lambda = 1$ to fermions. The physical meaning of this equation is the following: By assumption, the statistical weight remains to be *a single combinatoric number*, so one can count the states by thinking of the particles *effectively either as bosons or as fermions*, with the effective number of available single-particle states being *linearly dependent on the particle number*:

$$G_{eff}^{(b)} = G - \lambda(N-1), \quad \text{or} \quad G_{eff}^{(f)} = G - (1-\lambda)(N-1). \tag{3.3}$$

Obviously, for genuine bosons (or fermions), $G^{(b)}_{eff}$ (or $G^{(f)}_{eff}$) is independent of the particle number. In all other cases, either of the two G_{eff} is linearly dependent on the particle number. This is the defining feature of the FES. The statistics parameter λ tells us, on the average, how many single-particle states that a particle can exclude others to occupy. A proper understanding of this has been discussed in [27]. Thus, the expression (3.2) for the statistical weight, W, formulates a generalized Pauli exclusion principle, as first recognized by Haldane [2].

It is easy to generalize this state counting to more than one species, labeled by the index i:

$$W = \prod_i \frac{[G_i + N_i - 1 - \sum_j \lambda_{ij}(N_j - \delta_{ij})]!}{(N_i)!\,[G_i - 1 - \sum_j \lambda_{ij}(N_j - \delta_{ij})]!} . \tag{3.4}$$

Here G_i is the number of states when the system consists of only a single particle of species i. By definition, the diagonal λ_{ii} is the "self-exclusion" statistics of species i, while the non-diagonal λ_{ij} (for $i \neq j$) is the mutual-exclusion statistics. Note that λ_{ij}, which Haldane [2] called *statistical interactions*, may be *asymmetric* in i and j. The interpretation is similar to that of the one-species case: The number of available single-particle states for species i, in the presence of other particles, is again linearly dependent on particle numbers of all species:

$$G^{(b)}_{eff,i} = G_i - \sum_j \lambda_{ij}(N_j - \delta_{ji}),$$

or

$$G^{(f)}_{eff,i} = G^{(b)}_{eff,i} + N_i - 1. \tag{3.5}$$

The definition (3.2) or (3.4) starts with a postulated form for the statistical weight, and thus is more direct and convenient for the purpose of formulating quantum statistical mechanics. One of us [3] has first formulated the quantum statistical mechanics by proposing the notion of generalized ideal gas(GIG): A GIG satisfies the following two conditions: (i) The total energy (eigenvalue) is always of the form of a simple sum, in which the i-th term is linear in the particle number N_i:

$$E = \sum_i N_i \varepsilon_i^0, \tag{3.6}$$

with ε_i^0 identified as the energy of a particle of species i; (ii) The state-counting (3.4) for statistical weight W is applicable. When there are no statistical interactions (i.e., $\lambda_{ij} = 0$ for $i \neq j$), we have the usual ideal gas, which we call as IEG.

With the assumptions (3.6) and (3.4), the thermodynamics of a GIG can be worked out by the usual techniques in statistical mechanics. Consider a grand canonical ensemble at temperature T and with chemical potential μ_i for species i, whose partition function is given by

$$Z = \sum_{\{N_i\}} W(\{N_i\})\ \exp\{\sum_i N_i(\mu_i - \varepsilon_i^0)/T\} . \tag{3.7}$$

As usual, we expect that for very large N_i, the summation has a very sharp peak around the set of most-probable (or mean) particle numbers $\{\bar{N}_i\}$. Using the Stirling formula, introducing the average "occupation number per state" defined by $n_i \equiv \bar{N}_i/G_i$, and maximizing

$$\frac{\partial}{\partial n_i}\left[\ln W + \sum_i G_i n_i\,(\mu_i - \varepsilon_i^0)/T\right] = 0 , \tag{3.8}$$

one obtains the equations that determine the most-probable distribution of n_i

$$\sum_j (\delta_{ij} w_j + g_{ij}) n_j = 1 , \tag{3.9}$$

with $g_{ij} \equiv \lambda_{ij} G_j / G_i$, and w_i being determined by the functional equations

$$(1 + w_i) \prod_j \Big(\frac{w_j}{1 + w_j}\Big)^{\lambda_{ji}} = e^{(\varepsilon_i^0 - \mu_i)/T}. \tag{3.10}$$

The thermodynamic potential $\Omega = -T \ln Z$ and the entropy S are then given by

$$\begin{aligned}\Omega \equiv -PV &= -T \sum_i G_i \log \frac{1 + n_i - \sum_j g_{ij} n_j}{1 - \sum_j g_{ij} n_j} \\ &= -T \sum_i G_i \ln(1 + w_i^{-1}); \end{aligned} \tag{3.11}$$

$$\begin{aligned} S &= \sum_i G_i \Big\{ n_i \frac{\varepsilon_i^0 - \mu_i}{T} + \ln \frac{1 + n_i - \sum_j g_{ij} n_j}{1 - \sum_j g_{ij} n_j} \Big\} \\ &= \sum_i G_i \Big\{ n_i \frac{\varepsilon_i^0 - \mu_i}{T} + \ln(1 + w_i^{-1}) \Big\}. \end{aligned} \tag{3.12}$$

Other thermodynamic functions follow straightforwardly. As usual, one can easily verify that the fluctuations, $(\overline{N_i^2} - \bar{N}_i^2)/\bar{N}_i^2$, of the occupation numbers are negligible, which justifies the validity of the above approach.

IV. BOSONIZATION OF 1-D IDEAL EXCLUSON GAS

Let us first consider the simplest case, the 1-d IEG without internal degrees of freedom. We expect to obtain a continuous interpolation between the usual ideal Bose and ideal Fermi gas. Moreover, we want to show that the low-energy behavior of the IEG reproduces that of the Luttinger liquid and, therefore, provides a better characterization of the infrared fixed points associated with the Luttinger liquid.

A. Ideal Excluson Gas

Consider a GIG of N particles on a ring with size L. Single-particle states are labeled by pseudo-momenta k_i. The total energy and momentum are given by

$$E = \sum k_i^2, \qquad P = \sum k_i. \tag{4.1}$$

According to (3.5), in the thermodynamic limit the hole density, $\rho_a(k,T)$, (or the density of available single-particle states) is *linearly* dependent on the particle density, $\rho(k,T)$. By definition, the statistics interaction matrix is given by

$$\lambda(k_i,k_j) = -\frac{\Delta\rho_a(k_i)}{\Delta\rho(k_j)}. \tag{4.2}$$

Or in the thermodynamic limit, one has

$$\lambda(k,k') = -\,\delta\rho_a(k)/\delta\rho(k'). \tag{4.3}$$

The system is called an IEG of statistics λ (with *no mutual statistics* between different momenta), if

$$\lambda(k_i,k_j) = \lambda\,\delta(k_i-k_j), \tag{4.4}$$

or (3.5) reads

$$\rho(k_j) = \frac{1}{2\pi} + \frac{1}{L}(1-\lambda)\sum_{i\neq j}\delta(k_j-k_i)\rho(k_i)\Delta k, \tag{4.5}$$

which, in the thermodynamic limit, can be simply written as [5]

$$\rho_a(k,T) + \lambda\rho(k,T) = \rho_0(k,T), \tag{4.6}$$

where $\rho_0(k) \equiv 1/2\pi$ is the bare density of single-particle states. Thus, $\lambda = 1$ corresponds to fermions, and $\lambda = 0$ to bosons. The thermodynamic potential, now reads, in terms of (3.11)

$$\Omega = -\frac{T}{2\pi}\int_{-\infty}^{\infty} dk\,\ln(1+w(k,T)^{-1}), \tag{4.7}$$

with the function $w(k,T) \equiv \rho_a(k)/\rho(k)$ satisfying an algebraic equation,

$$w(k,T)^{\lambda}[1+w(k,T)]^{1-\lambda} = e^{(k^2-\mu)/T}. \tag{4.8}$$

Firstly, we consider the ground state, in which the particles are distributed in a finite and origin-symmetric interval in the pseudo-momentum space. The (pseudo-)Fermi momentum is defined by

$$k_F^2 = \mu \tag{4.9}$$

and its value is fixed by the average particle density $\bar{d}_0 = N_0/L$ in the ground state,

$$\int_{-k_F}^{k_F} dk\rho(k) = \bar{d}_0. \tag{4.10}$$

Because holes are absent in the ground state, the particle density in the ground state is easily obtained from (4.6),

$$\rho(k) = \begin{cases} \dfrac{1}{2\pi\lambda}, & \text{for } |k| < k_F; \\ 0, & \text{for } |k| > k_F. \end{cases} \tag{4.11}$$

Hence, one has

$$k_F = \pi\lambda\bar{d}_0, \qquad \mu = (\pi\lambda\bar{d}_0)^2. \tag{4.12}$$

Then the ground state energy and momentum are given by

$$\frac{E_0}{L} = \int_{-k_F}^{k_F} dk\rho(k)k^2 = \frac{1}{3}\pi^2\lambda^2\bar{d}_0^3,$$
$$P_0 = \int_{-k_F}^{k_F} dk\rho(k)k = 0. \tag{4.13}$$

Now let us examine possible excitations in an IEG. First there are density fluctuations due to particle-hole excitations, i.e., sound waves with velocity (see the next subsection)

$$v_s = v_F \equiv 2k_F. \tag{4.14}$$

Besides, by adding extra M particles to the ground state, one can create particle excitations, and by Galileo boosts a persistent current. We observe that the velocities of these three classes of elementary excitations in IEG also satisfy the fundamental relation (2.1). Indeed, shifting N_0 to $N = N_0 + M$, the change in the ground state energy is

$$\delta_M E_0 = \frac{1}{3}\pi^2\lambda^2(N/L)^3 - \frac{1}{3}\pi^2\lambda^2(N_0/L)^3$$
$$= \pi^2\bar{d}_0^2 M + \pi(\lambda k_F)M^2 + O(M^3/L^3), \tag{4.15}$$

while a persistent current, created by the boost of the Fermi sea $k \to k + \pi J/L$, leads to the energy shift

$$\delta_J E_0 = \int_{-k_F+\pi J/L}^{k_F+\pi J/L} dk\rho(k)k^2 - \int_{-k_F}^{k_F} dk\rho(k)k^2$$
$$= \pi(k_F/\lambda)J^2. \tag{4.16}$$

Therefore the total change in energy, due to charge and current excitations, is

$$\delta E_0 - \mu M = \frac{\pi}{2L}v_F(\lambda M^2 + \lambda^{-1}J^2). \tag{4.17}$$

The total momentum change due to the current excitations is

$$\delta P_0 = \sum_k \frac{\pi J}{L} = \pi(\bar{d}_0 + \frac{M}{L})J. \tag{4.18}$$

If we denote the variation in free energy as $\delta F_0 = \delta E_0 - \mu M$, and identify λ as the controlling parameter $e^{-2\varphi}$ in the Luttinger liquid theory, (4.17) just recuperates the zero-mode contributions [28] in (2.4). Comparing (4.17) with (2.4) we identify the velocities v_N and v_J to be

$$v_N = v_F \lambda, \qquad v_J = v_F/\lambda, \tag{4.19}$$

Then we see the velocity relation (2.1), i.e., $v_s = \sqrt{v_N v_J}$, that Haldane used to characterize the Luttinger liquids, is satisfied in IEG. The selection rule (2.5) also holds for the IEG, since the system should correspond to the ideal Fermi gas if $\lambda = 1$.

Encouraged by this relationship between the IEG and Luttinger liquids, we want to calculate the critical exponents of IEG to see whether they reproduce those of the Luttinger liquids. This motivates to develop a bosonization for the density fluctuations in IEG.

B. Low Energy Limit and Bosonization

Following Yang and Yang [4,18], we introduce the dressed energy $\epsilon(k,T)$ by writing

$$w(k,T) = e^{\epsilon(k,T)/T}. \tag{4.20}$$

The point is that the grand partition function Z_G, corresponding to the thermodynamic potential (4.7), is of the form of that for an ideal system of fermions with a complicated, T-dependent energy dispersion given by the dressed energy:

$$Z_G = \prod_k (1 + e^{-\epsilon(k,T)/T}). \tag{4.21}$$

However, this fermion representation is not very useful, because of the implicit T-dependence of the dressed energy. To simplify, we consider the low-T limit. By using the dressed energy, (4.8) reads

$$\epsilon(k,T) = k^2 - \mu - T(1-\lambda)\ln(1 + e^{-\epsilon(k,T)/T}). \tag{4.22}$$

Because there is no singularity in $\epsilon(k,T)$ at $T = 0$, the zero temperature dressed energy is given by

$$\epsilon(k) = \begin{cases} (k^2 - k_F^2)/\lambda, & |k| < k_F, \\ k^2 - k_F^2, & |k| > k_F. \end{cases} \tag{4.23}$$

Denote

$$\epsilon(k,T) = \epsilon(k) + \tilde{\epsilon}(k,T), \tag{4.24}$$

where

$$\tilde{\epsilon}(k,0) = 0. \tag{4.25}$$

In the low-T limit, one has

$$\epsilon(k,T) = \begin{cases} \frac{k^2-\mu}{\lambda} - (\lambda^{-1} - 1)T\ln(1 + e^{-|\epsilon(k)|/T}), & |k| < k_F, \\ (k^2 - \mu) - (1-\lambda)T\ln(1 + e^{-|\epsilon(k)|/T}), & |k| > k_F, \end{cases} . \tag{4.26}$$

Hence,

$$\tilde{\epsilon}(k) = \begin{cases} (1-\lambda^{-1})T\ln(1 + e^{-|\epsilon(k)|/T}), & |k| < k_F, \\ (\lambda - 1)T\ln(1 + e^{-|\epsilon(k)|/T}), & |k| > k_F. \end{cases} \tag{4.27}$$

For low energies, one can consider only the excitations around the Fermi surface,

$$\begin{aligned} \frac{\Omega(T)}{L} &\approx -\frac{T}{2\pi}\int_{-\infty}^{\infty} dk \ln(1 + e^{-\epsilon(k)/T}) \\ &\quad + \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \frac{\tilde{\epsilon}(k,T)}{1 + e^{\epsilon(k)/T}} \\ &\approx \frac{1}{2\pi}\int_{-k_F}^{k_F} dk\epsilon(k) - \frac{T}{\pi\lambda}\int_{k_F-\delta}^{k_F} dk \ln(1 + e^{-|\epsilon(k)|/T}) \\ &\quad - \frac{T}{\pi}\int_{k_F}^{k_F+\delta} dk \ln(1 + e^{-|\epsilon(k)|/T}), \end{aligned} \tag{4.28}$$

where the first term on the right hand side of the last equality is recognized as $\Omega(0)/L$. The cut-off δ is of order $O(T/v_s)$ (actually, a few times of T/v_s). Mathematically, we take the limit of $T \to 0$ followed by $\delta \to 0$. Using the integral formula

$$\int_0^\infty dx \ln(1 + e^{-x}) = \frac{\pi^2}{12},$$

we have the low-T thermodynamic potential

$$\frac{\Omega(T)}{L} - \frac{\Omega(0)}{L} = -\frac{\pi T^2}{6 v_s}, \tag{4.29}$$

which implies that the theory is cut-off independent at low temperature. Notice that $F = \Omega - \mu N$. Because we only consider the particle-hole excitations near the Fermi surface contribute to thermal excitations, $N(T) - N(0) = 0$, which can be checked by an explicit calculation in terms of the definition of $\rho(k,T)$. Thus, we have

$$\frac{F(T)}{L} - \frac{F(0)}{L} = \frac{\Omega(T)}{L} - \frac{\Omega(0)}{L} = -\frac{\pi T^2}{6 v_s}. \tag{4.30}$$

This means that the low energy behavior of the IEG is controlled by a $c = 1$ CFT. This result can be verified by a finite-size scaling in the spatial direction,

$$\frac{F_L(0)}{L} - \frac{F(0)}{L} = -\frac{\pi v_s}{6 L^2}, \tag{4.31}$$

where $F_L(0)$ is the zero temperature free energy for a system with size L. (For details, see [9].)

The above relation agrees with the finite-size scaling of a conformally invariant system with central charge $c =$

1. So we want to see whether the low-energy effective theory of the IEG is really a CFT. Let us start with the grand partition function (4.21). At low temperature, the solution (4.26) leads to $\tilde{\epsilon}(k,T) = O(Te^{-|\epsilon|/T})$, so one can simply replace $\epsilon(k,T)$ with $\epsilon(k)$ in the grand partition function:

$$Z_G \simeq \prod_k (1 + e^{\beta\epsilon(k)}). \tag{4.32}$$

Note that the dressed energy with k outside the Fermi points $\pm k_F$ has a slope different from that with k inside $\pm k_F$. The former is $\frac{2\pi}{L}$ and the latter $\frac{2\pi\lambda}{L}$. It is necessary to keep this in mind for writing down the correct ground state wave functions of the excluson system. Now that the dispersion $\epsilon(k)$ is T-independent, the grand partition function in the low-T limit can be expressed in a fermionic representation as

$$Z_G = \mathrm{Tr} e^{-\beta H_{\rm eff}}, \tag{4.33}$$

where the effective Hamiltonian is given by

$$H_{\rm eff} = \sum_k \epsilon(k)\, c_k^\dagger c_k, \tag{4.34}$$

where $c_k^\dagger$ are fermionic creation operators. We also see that $\epsilon(k_F) = 0$, which can be used to define the Fermi momentum.

Physically, it is the phonon excitations that dominate the low-energy behavior of the system. In the low-T limit, it is enough to consider the density fluctuations only near the Fermi points, $k \sim \pm k_F$, where the left- and right-moving sectors are separable and decoupled:

$$H_{\rm eff} = H_+ + H_-. \tag{4.35}$$

Besides this, another important simplification for excitations near Fermi points in the low-T limit is that their energy, $H_\pm$, has a linearized dispersion:

$$\epsilon_\pm(k) = \begin{cases} \pm v_F(k \mp k_F), & |k| > k_F, \\ \pm v_F(k \mp k_F)/\lambda, & |k| < k_F. \end{cases} \tag{4.36}$$

We note the 'refractions' at $k = \pm k_F$, which implies to create a particle with pseudo-momentum k and to create a hole with k' cost different energies, even if $|k - k_F| = |k' - k_F|$. The reason for this is that k is not the actual momentum carried by $c_k^\dagger$, as we will see soon.

The key thing for bosonization is to construct a density fluctuation operator. Taking into account the different slopes for dressed energy inside and outside the Fermi points, the density fluctuation operator at $k \sim k_F$ is constructed as follows:

$$\rho_q^{(+)} = \sum_{k>k_F} :c_{k+q}^\dagger c_k: + \sum_{k<k_F-\lambda q} :c_{k+\lambda q}^\dagger c_k: + \sum_{k_F-\lambda q<k<k_F} :c^\dagger_{\frac{k-k_F}{\lambda}+k_F+q} c_k: \tag{4.37}$$

for $q > 0$. A similar density operator $\rho_q^{(-)}$ can also be defined at $k \sim -k_F$,

$$\rho_q^{(-)} = \sum_{k<-k_F} :c_{k-q}^\dagger c_k: + \sum_{k>-k_F+\lambda q} :c_{k-\lambda q}^\dagger c_k: + \sum_{-k_F+\lambda q>k>-k_F} :c^\dagger_{\frac{k+k_F}{\lambda}-k_F-q} c_k: \tag{4.38}$$

To define the normal ordering we write, e.g.,

$$c_k = \begin{cases} c_k, & k > k_F, \\ d_k^\dagger, & k < k_F, \end{cases} \tag{4.39}$$

where $d_k^\dagger$ is understood as a creation operator of a hole. Then normal ordering is done as usual: putting the annihilation operators to the right of the creation ones. Hence we have, e.g.,

$$\rho_q^{(+)} = \sum_{k>k_F} :c_{k+q}^\dagger c_k: + \sum_{k<k_F-\lambda q} :d_{k+\lambda q} d_k^\dagger: + \sum_{k_F-\lambda q<k<k_F} :c^\dagger_{\frac{k-k_F}{\lambda}+k_F+q} d_k^\dagger: \tag{4.40}$$

Within the Tomonaga approximation [29], in which commutators are taken to be their ground-state expectation value, we obtain

$$\begin{aligned} [\rho_q^{(\pm)}, \rho_{q'}^{(\pm)\dagger}] &\approx \langle 0|[\rho_q^{(\pm)}, \rho_q^{(\pm)\dagger}]|0\rangle \\ &= \sum_{k_F-\lambda q<k<k_F} \langle 0|c_{k+\lambda q} c^\dagger_{k+\lambda q'}|0\rangle \\ &= \delta_{q,q'} \sum_{k_F-\lambda q<k<k_F} 1 = \frac{L}{2\pi} q\delta_{q,q'} \end{aligned} \tag{4.41}$$

Also, the commutators between $H_{\rm eff}$ and $\rho_q^{(\pm)}$ are

$$[H_\pm, \rho_q^{(\pm)}] \approx \langle 0|[H_\pm, \rho_q^{(\pm)}]|0\rangle = \pm v_F q \rho_q^{(\pm)}. \tag{4.42}$$

(4.41) and (4.42) describe 1-d free phonons with the sound velocity $v_s = v_F$ (so we have proved (4.14)). Introducing normalized boson annihilation operators

$$b_q = \sqrt{2\pi/qL}\,\rho_q^{(+)}, \qquad \tilde{b}_q = \sqrt{2\pi/qL}\,\rho_q^{(-)\dagger} \tag{4.43}$$

and adding back the zero mode contributions, the bosonized Hamiltonian satisfying (4.41) is given by

$$H_B = v_s\{\sum_{q>0} q(b_q^\dagger b_q + \tilde{b}_q^\dagger \tilde{b}_q) + \frac{1}{2}\frac{\pi}{L}[\lambda M^2 + \frac{1}{\lambda}J^2]\}, \tag{4.44}$$

which agrees with the bosonized Hamiltonian (2.4) in the Luttinger liquid theory.

In passing, we make a comment on linearization of the dressed energy dispersion. When we did this, we changed the ground state energy, because we assumed that for all k the spectrum is linear in k. However, we changed neither the ground state wave function, nor the low-T physics. On the other hand, the linearized spectrum was valid only for phonon excitations, it has nothing to do with the zero-mode excitations. So, after the linearized phonon part of the Hamiltonian is bosonized, we had to add back the zero-mode excitations.

The construction of the bosonized momentum operator is a bit more tricky, because $c_k^\dagger$ does not carry a momentum k. Each term in (4.37) should carry the same momentum q, therefore the fermion created by $c_k^\dagger$ carries a dressed momentum p, which is related to k by

$$p(k) = \begin{cases} k - k_F + (k_F/\lambda), & k > k_F, \\ k/\lambda, & |k| < k_F, . \\ k + k_F - (k_F/\lambda), & k < -k_F. \end{cases} \tag{4.45}$$

In terms of this variable, the linearized dressed energy $\epsilon(p)$ is of a simple form: $\epsilon_\pm(p) = \pm v_s(p \mp p_F)$, with $p_F = k_F/\lambda$. The bosonized total momentum operator, corresponding to the fermionized $P = \sum_k p(k)\, c_k^\dagger c_k$, is

$$P = \sum_{q>0} q(b_q^\dagger b_q - \tilde{b}_q^\dagger \tilde{b}_q) + \pi(\bar{d}_0 + M/L)\, J. \tag{4.46}$$

We see that the fundamental velocity relation, the bosonized Hamiltonian and momentum, and the selection rule of the quantum numbers in the Luttinger liquid theory can all be reproduced in IEG if we identify

$$\lambda \equiv e^{-2\varphi}. \tag{4.47}$$

To say that IEG can be used to characterize the renormalization group *fixed points* of Luttinger liquids, we still need to check the conformal invariance of the bosonized theory of IEG, and to verify the critical properties of IEG reproduce those of the Luttinger liquids.

C. Effective Field Theory and Conformal Invariance

To check conformal invariance, we need to rewrite the above bosonized effective Halmitonian (4.44) into a form of field theory in coordinate space. Employing the Fourier transformation, the density operator can be written as

$$\begin{aligned} \rho(x) &= \rho_R(x) + \rho_L(x), \\ \rho_R(x) &= \frac{M_R}{L} + \sum_{q>0} \sqrt{\frac{q}{2\pi L\lambda}} (e^{iqx} b_q + e^{-iqx} b_q^\dagger), \\ \rho_L(x) &= \frac{M_L}{L} + \sum_{q>0} \sqrt{\frac{q}{2\pi L\lambda}} (e^{-iqx} \tilde{b}_q + e^{iqx} \tilde{b}_q^\dagger), \end{aligned} \tag{4.48}$$

where $M_{R,L}$ are given by $M = M_R + M_L$ and $\tilde{b}_q = b_{-q}$ for $q > 0$.

The boson field $\phi(x)$, which is conjugated to $\rho(x)$ and satisfies

$$[\phi(x), \rho(x')] = i\delta(x - x'), \tag{4.49}$$

is given by

$$\begin{aligned} \phi(x) &= \phi_R(x) + \phi_L(x), \\ \phi_R(x) &= \frac{\phi_0}{2} + \frac{\pi J_R x}{L} + i \sum_{q>0} \sqrt{\frac{\pi\lambda}{2qL}} (e^{iqx} b_q - e^{-iqx} b_q^\dagger), \\ \phi_L(x) &= \frac{\phi_0}{2} + \frac{\pi J_L x}{L} + i \sum_{q>0} \sqrt{\frac{\pi\lambda}{2qL}} (e^{-iqx} \tilde{b}_q - e^{iqx} \tilde{b}_q^\dagger), \end{aligned} \tag{4.50}$$

with $J = J_R + J_L$. We have to assign the quantum numbers such that there are only two independent each other in $M_{R,L}$ and $J_{R,L}$. A consistent choice is

$$M_R = J_R, \qquad M_L = -J_L. \tag{4.51}$$

Then,

$$J = J_R + J_L, \qquad M = J_R - J_L. \tag{4.52}$$

Here ϕ_0 is an angular variable conjugated to M: $[\phi_0, M] = i$. The Hamiltonian (2.4) becomes

$$H = \frac{1}{2} \int_0^L dx\, [\pi v_N \rho(x)^2 + v_J/\pi\, (\partial_x \phi(x))^2], \tag{4.53}$$

or by a field rescaling,

$$H = \frac{v_s}{2\pi} \int_0^L dx\, [\Pi(x)^2 + (\partial_x X(x))^2], \tag{4.54}$$

where

$$\Pi(x) = \pi\lambda^{1/2} \rho(x), \qquad X(x) = \lambda^{-1/2} \phi(x). \tag{4.55}$$

With $X(x,t) = e^{iHt} X(x) e^{-iHt}$, the Lagrangian density reads

$$\mathcal{L} = \frac{v_s}{2\pi}\, \partial_\alpha X(x,t)\, \partial^\alpha X(x,t). \tag{4.56}$$

This is the Lagrangian density of a free scalar field theory in $1+1$-dimensions. Writing the corresponding operators as the functionals of the scalar field $X(x,t)$, all correlation functions can be obtained by using the propagators of $X_R(x,t)$ and $X_L(x,t)$,

$$\begin{aligned} \langle X_R(x,t) X_R(0,0) \rangle &= -\frac{1}{4} \ln(x - v_s t), \\ \langle X_L(x,t) X_L(0,0) \rangle &= -\frac{1}{4} \ln(x + v_s t). \end{aligned} \tag{4.57}$$

The statistics of an operator in the theory can also be inferred by the commutators of the scalar fields,

$$[X_{R,L}(x), X_{R,L}(x')] = \pm \frac{i\pi}{4} \theta(x - x'). \tag{4.58}$$

We recognize that $\mathcal{L}$ (4.56) is the Lagrangian of a $c = 1$ CFT [19], consistent with the finite-size scaling (4.30). Alternatively, it is easy to check that the theory is invariant under the conformal transformations generated by a set of the Virasoro generators

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty}\alpha_{n-m}\alpha_n, \qquad \tilde{L}_m = \frac{1}{2}\sum_{n=-\infty}^{\infty}\tilde{\alpha}_{n-m}\tilde{\alpha}_n, \tag{4.59}$$

where the oscillators $\alpha_m = m^{1/2}b_q$ and $\tilde{\alpha}_{-m} = m^{1/2}\tilde{b}_q^\dagger$ for $m = qL/2\pi > 0$ being integers. $\alpha_0 = (\pi/2L)^{1/2}[J\lambda^{-1/2} - M\lambda^{1/2}]$ and $\tilde{\alpha}_0 = (\pi/2L)^{1/2}[J\lambda^{-1/2} + M\lambda^{1/2}]$. The generators obey the Virasoro algebra with the central charge $c = 1$,

$$[L_m^{\text{tot}}, L_n^{\text{tot}}] = (m-n)L_{m+n}^{\text{tot}} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}, \tag{4.60}$$

with $L_m^{\text{tot}} = L_m + \tilde{L}_m$.

Since ϕ_0 is an angular variable, there is a hidden invariance in the theory under $\phi \to \phi + 2\pi$. The field X is thus said to be "compactified" on a circle, with a radius that is determined by the exclusion statistics [9,21]:

$$X \sim X + 2\pi R, \quad R^2 = 1/\lambda. \tag{4.61}$$

Noting the selection rule (2.5), the Hamiltonian has a duality

$$\lambda \leftrightarrow 1/\lambda, \qquad M \leftrightarrow J, \tag{4.62}$$

which has referred to the particle-hole duality [5,6]. Using the CFT terminology, this duality is represented as the duality of the compactified radii,

$$R \leftrightarrow 1/R. \tag{4.63}$$

We note that this is different from the duality relation $R \leftrightarrow 2/R$, in the usual $c = 1$ CFT [19] compactifed on a circle. Actually, according to the standard terminology in CFT [19], our selection rule (2.5) and duality relation (4.63) make what we obtained above a $c = 1$ CFT compactified on an *orbifold* S^1/Z_2, i.e., a circle folded by a reflection about a diameter, which topologically is a semi-circle or an interval. This difference can also be seen from the grand partition function: Using the identification between H_{eff} and L_0^{tot}, i.e., $H_{\text{eff}} = v_s L_0^{\text{tot}}$, the grand partition function of IEG (in the low-T limit) can be rewritten as

$$Z_G = Tr_{\mathcal{H}}[q^{L_0}\bar{q}^{\tilde{L}_0}], \tag{4.64}$$

where $q = e^{iv_s\tau}$ with $\tau = i\beta = i/T$. Thus, the selection rule (2.5) severely constrain the allowed values for the eigenvalues of L_0 and $\tilde{L}_0$. It makes the CFT we obtained have an unusual spectrum and duality relation, corresponding to the $c = 1$ orbifold CFT [19]. In next subsection we will see that because of the difference in the selection rules, the statistics of the allowed charge-1 operators in the two classes of CFT's are not the same.

We note that a similar situation happens for the CFT that describes the low-T behavior of the Calogero-Sutherland (C-S) model [17,18]. This model has two different versions, with the long-range interactions being among bosons or among fermions, respectively. At low temperature, the two versions have different selection rules for the zero-mode quantum numbers, thus leading to different CFT's: The low-T CFT for the bosonic C-S model is the usual $c = 1$ CFT compactified on a circle, which has been studied extensively in the literatures [20–24,9]; while for the fermionic C-S model the low-T limit gives rise to the $c = 1$ orbifold CFT. This is because the selection rule for zero modes severely constrains the spectrum of the system, i.e., possible quantum numbers of the allowed excitations. (For details, see ref. [9].) Thus, only the fermionic (not the bosonic) C-S model respects a duality relation $\lambda \leftrightarrow 1/\lambda$ that coincides with the particle-hole duality in IEG [5,6].

D. Correlation Functions

The CFT description of the IEG offers a better understanding for the space of quantum states in the theory. States $V[X]|0\rangle$ or operators $V[X]$ are *allowed* only if they respect the invariance (4.61),

$$V[X + 2\pi R] \equiv V[X], \tag{4.65}$$

with a given boundary condition restriction. Here, a Fermi or a Bose operator obeys the periodic boundary condition (PBC). So quantum numbers of quasiparticles are strongly constrained, in particular by the selection rule for zero-mode quantum numbers. For example, the primary fields obeying the PBC in the CFT are given by

$$\begin{aligned}\phi_{M,J}(x) &\sim f(J, X^0) : e^{i(M\lambda^{1/2}+J/\lambda^{1/2})X_R(x)} \\ &\quad\times e^{i(M\lambda^{1/2}-J/\lambda^{1/2})X_L(x)} :, \\ f(J, X^0) &= e^{iJ(\lambda^{1/2}-\lambda^{-1/2})X_R^0}e^{-iJ(\lambda^{1/2}-\lambda^{-1/2})X_L^0}\end{aligned} \tag{4.66}$$

where the prefactor $f(J, X^0)$ makes the fields satisfy the PBC, M and J eigenvalues of the number and current operators, and $X^0 = \pi Mx/L$. The field carries the charge M and current J. The conformal dimensions of the fields are

$$\begin{aligned}h &= \frac{1}{2}[(M\lambda^{1/2} + J/\lambda^{1/2})^2 + (M\lambda^{1/2} - J/\lambda^{1/2})^2] \\ &= M^2\lambda + J^2\lambda^{-1}.\end{aligned} \tag{4.67}$$

The statistics of the field can be calculated by using (4.58) and the statistics factors are

$$\exp\{i\frac{\pi}{4}[(M\lambda^{1/2}+J/\lambda^{1/2})^2-(M\lambda^{1/2}-J/\lambda^{1/2})^2]\}$$
$$=(-1)^{MJ}. \tag{4.68}$$

Consider the charge-1 primary fields, with $M = 1$. Therefore, they can only be fermions since $J =$ odd due to the selection rule. The general charge-1 fermion operator is a linear combination of the charge-1 primary fields. A careful construction of the allowed fermion field with unit charge leads to

$$\Psi_F^\dagger(x,t)=\rho(x)^{1/2}\sum_{m=-\infty}^{\infty}e^{iO_m}:e^{i(\lambda^{1/2}+(2m+1)/\lambda^{1/2})X_R(x_-)}:$$
$$:e^{i(\lambda^{1/2}-(2m+1)/\lambda^{1/2})X_L(x_+)}:, \tag{4.69}$$

where the prefactor f has been suppressed and the hermitian, constant-valued operators O_m satisfy [30]

$$[O_m,O_{m'}]=i\pi(m-m'). \tag{4.70}$$

The multi-sector density operator is the linear combination of those primary fields with $M = 0$ and $J =$even,

$$\hat\rho(x)=\Psi_F^\dagger(x)\Psi_F(x)$$
$$=\rho(x)\sum{}_m:\exp\{i2m[X_R(x)-X_L(x)]/\lambda^{1/2}\}:. \tag{4.71}$$

All the secondary fields in the CFT follow by considering the sound wave contribution to the conformal weight of the fields.

The correlation functions can easily be calculated by using the CFT techniques. For examples, the density-density and single particle correlation functions are as follows,

$$\langle\hat\rho(x,t)\hat\rho(0,0)\rangle\approx\bar d_0^2\Bigg[1+\frac{1}{(2\pi\bar d_0)^2\lambda}\left(\frac{1}{x_R^2}+\frac{1}{x_L^2}\right)$$
$$+\sum_{m=1}^{\infty}A_m\frac{1}{[x_Rx_L]^{m^2/\lambda}}\cos(2\pi\bar d_0mx)\Bigg], \tag{4.72}$$

and

$$G(x,t)\equiv\langle\Psi_F^\dagger(x,t)\Psi_F(0,0)\rangle$$
$$\approx\bar d_0\sum_{m=-\infty}^{\infty}B_m\frac{1}{x_R^{(\lambda^{1/2}+(2m+1)\lambda^{-1/2})^2/4}}$$
$$\frac{1}{x_L^{(\lambda^{1/2}-(2m+1)\lambda^{-1/2})^2/4}}$$
$$e^{i(2\pi(m+\lambda/2)\bar d_0x+\mu t)}, \tag{4.73}$$

where $x_{R,L} = x \mp v_s t$ and A_m and B_m regularization-dependent constants.

Usually a physical quantity, e.g., a boson field, satisfies the periodic boundary conditions (PBC). Hence, a charge-1 bosonic excitations are not allowed in the theory, because it is anti-periodic. However, as we know, an anyon field needn't to obey the PBC. So in the theory, there may be allowed anyonic excitations. A charge-1 anyonic (or exclusonic) operator is a primary field that does not obey the PBC,

$$\Psi_\lambda^\dagger(x)=:\Psi_F^\dagger(x)e^{i(\lambda^{1/2}-\lambda^{-1/2})(X_R(x)-X_L(x))}:. \tag{4.74}$$

The anyon commutation relation is easy to check:

$$\Psi_\lambda^\dagger(x)\Psi_\lambda^\dagger(x')-e^{i\pi\lambda\mathrm{sgn}(x-x')}\Psi_\lambda^\dagger(x')\Psi_\lambda^\dagger(x)=0,\quad \text{for } x\neq x'. \tag{4.75}$$

In other words, the anyon field carries a fractional current. Or by the $M \leftrightarrow J$-duality, the anyon with integer J carries a fractional charge. The correlation function of the single-anyon reads

$$G(x,t;\lambda)\equiv\langle\Psi_\lambda^\dagger(x,t)\Psi_\lambda(0,0)\rangle \tag{4.76}$$
$$\approx\bar d_0\sum_{m=-\infty}^{\infty}B_m^a\frac{1}{x_R^{(m+\lambda)^2/\lambda}}\frac{1}{x_L^{m^2/\lambda}}e^{i(2\pi(m+\lambda/2)x+\mu t)}, \tag{4.77}$$

This correlation function coincides with the asymptotic one [7] in the Calogero-Sutherland model. We see that
(i) if $\lambda = 1$, (4.77) consists with (4.73);
(ii) there are no boson excitations ($\lambda = 0$) because $G(x, t; 0) = 0$;
(iii) and moreover, $\lambda > 0$ is implied since (4.77) will diverge at the long distance if $\lambda < 0$.
(iv) Look at $m = 0$. The critical exponents can be reads out,

$$\eta_f=\lambda+\lambda^{-1},\qquad \eta_\lambda=2\lambda.$$

Thus,

$$\eta_f>\eta_\lambda,\ \text{ if }\lambda<1;$$
$$\eta_f<\eta_\lambda,\ \text{ if }\lambda>1.$$

(v) The multi-sector density operator for exclusons is the same as that of the fermion.

The single-hole state, i.e. $\Psi_{1/\lambda}^\dagger|0\rangle\equiv\Psi_\lambda(\lambda\to\lambda^{-1})|0\rangle$, with charge $-1/\lambda$ alone is not allowed. The minimum allowed multi-hole state is given by

$$\Psi_{1/\lambda}^\dagger(x_1)...\Psi_{1/\lambda}^\dagger(x_p)|0\rangle$$

if $\lambda = p/q$ is rational. One may obtain, e.g. ,

$$\langle[\Psi_{1/\lambda}^\dagger(x,t)]^p[\Psi_{1/\lambda}(0,0)]^p\rangle\sim[G(x,t;1/\lambda)]^p.$$

A more interesting allowed operator is what creates q particle excitations accompanied by p hole excitations:

$$\hat n(x,t)=[\Psi_\lambda^\dagger(x,t)]^q[\Psi_{1/\lambda}^\dagger(x,t)]^p.$$

We note the similarity of this operator to Read's order parameter [31] for fractional quantum Hall fluids (in bulk). Its correlation function can be calculated by using Wick's theorem:

$$\langle \hat{n}(x,t)\hat{n}(0,0)\rangle \sim [G(x,t;\lambda)]^q [G(x,t;1/\lambda)]^p. \quad (4.78)$$

If the contribution from the $m = 0$ sector dominates, then one gets

$$\langle \hat{n}(x,t)\hat{n}(0,0)\rangle \sim (x - v_s t)^{-(p+q)}$$

.

V. TWO EXTENSIONS

Now we proceed to go beyond IEG. Two extensions will be discussed in this section: The one-component GIG with the mutual statistics, and the non-ideal gas with the Luttinger-type interactions. In either case, we will show that the low-temperature behavior is that of an IEG, controlled by a single "effective statistics" parameter λ_{eff}, whose value depends on the mutual statistics and the coupling constants in the interactions.

A. Generalized ideal gas with mutual statistics

We turn to discussing the effects of mutual statistics. Consider a GIG with the statistics matrix (4.3) in momentum space given by

$$g(k-k') = \delta(k-k') + \Phi(k-k'). \quad (5.1)$$

Here $\Phi(k) = \Phi(-k)$ is a smooth function. $\Phi(k-k')$ stands for mutual statistics between particles with different momenta; for IEG $\Phi(k) = (\lambda - 1)\delta(k)$. The thermodynamic properties of GIG is given by eq. (4.7), but now $w(k,T)$ satisfies integral equation [3,5] which, in terms of the dressed energy (4.20), is of the form

$$\epsilon(k,T) = \epsilon_0(k) + T\int_{-\infty}^{\infty}\frac{dk'}{2\pi}\,\Phi(k-k')\,\ln(1+e^{-\epsilon(k',T)/T})\,, \quad (5.2)$$

where $\epsilon_0(k) \equiv k^2 - \mu$. In the low-$T$ limit, it can be proven by the iteration [9] that $\epsilon(k,T) = \epsilon(k) + O(T^2/v_s)$, where $\epsilon(k)$ is the zero-temperature dressed energy given below. At $T = 0$, the Fermi momentum k_F is determined by

$$\epsilon(\pm k_F) = 0. \quad (5.3)$$

Introduce

$$(\alpha\cdot\beta)[-k_F,k_F] \equiv \int_{-k_F}^{k_F}\frac{dk}{2\pi}\,\alpha(k)\,\beta(k)\,, \quad (5.4)$$

$$(\Phi\cdot\alpha)(k;-k_F,k_F] \equiv \int_{-k_F}^{k_F}\frac{dk'}{2\pi}\,\Phi(k-k')\,\alpha(k')\,. \quad (5.5)$$

Then both $\rho(k)$ and $\epsilon(k)$ in the ground state satisfy an integral equations like

$$\alpha(k) = \alpha_0(k) - (\Phi\cdot\alpha)\,(k;-k_F,k_F]\,. \quad (5.6)$$

The dressed momentum $p(k)$ is related to $\rho(k)$ by

$$dp(k) = 2\pi\rho(k)dk, \qquad p(k) = -p(-k). \quad (5.7)$$

The ground state energy is given by

$$E_0/L = (\epsilon_0\cdot\rho)[-k_F,k_F]. \quad (5.8)$$

Using the equation satisfied by $\rho(k)$, it can be expressed by the dressed energy

$$E_0/L = (\epsilon\cdot\rho_0)[-k_F,k_F]. \quad (5.9)$$

The above equations are of the same form as those in the thermodynamic Bethe ansatz [4], hence the Luttinger-liquid relation [32], $v_s = \sqrt{v_N v_J}$, remains true. A simple proof is sketched as follows. The sound velocity is well-known:

$$v_s = \partial\epsilon(p_F)/\partial p_F. \quad (5.10)$$

The charge velocity is given by

$$v_N = v_s\,z(k_F)^{-2}, \quad (5.11)$$

where the dressed charge $z(k)$ [32] is given by the solution to the integral equation

$$z(k) = 1 - (\Phi\cdot z)(k;-k_F,k_F]. \quad (5.12)$$

This relation can be easily derived from the definitions

$$\begin{aligned} v_N &= L\partial\mu/\partial N_0, \\ z(k) &= -\delta\epsilon(k)/\delta\mu. \end{aligned} \quad (5.13)$$

To create a persistent current, let us boost the Fermi sea by

$$\pm k_F \to \pm k_F + \Delta, \quad (5.14)$$

where $\Delta = z(k_F)/L\rho(k_F)$. Then the total energy of the state with the persistent current is

$$\begin{aligned} E_\Delta/L &= (\epsilon_0\cdot\rho_\Delta)[-k_F+\Delta,k_F+\Delta] \\ &= (\epsilon_\Delta\cdot\rho_0)[-k_F+\Delta,k_F+\Delta], \end{aligned} \quad (5.15)$$

where

$$\rho_\Delta(k) = \rho_0(k) - (\Phi\cdot\rho_\Delta)(k;-k_F+\Delta,k_F+\Delta] \quad (5.16)$$

and

$$\epsilon_\Delta(k) = \epsilon_0(k) - (\Phi\cdot\epsilon_\Delta)(k;-k_F+\Delta,k_F+\Delta]. \quad (5.17)$$

Now, using the last expression for E_Δ and substituting ϵ_Δ in (V A), we have

$$E_\Delta/L = (\epsilon\cdot\rho_0)[-k_F,k_F] + \frac{\Delta^2}{2}\epsilon'(k_F)$$
$$\{\rho_0(k_F) + (\rho_0\cdot 2\pi F)(k_F;-k_F,k_F]\}$$
$$-\frac{\Delta^2}{2}\epsilon'(-k_F)\{\rho_0(-k_F) + (\rho_0\cdot 2\pi F)(-k_F;-k_F,k_F]\}. \quad (5.18)$$

Here $F(k,k')$ is determined by

$$F(k,k') = \frac{1}{2\pi}\Phi(k,k') - \frac{1}{2\pi}\int_{-k_F}^{k_F} dk''\Phi(k,k'')F(k'',k'). \quad (5.19)$$

On the other hand, we note that the equation for $\rho_0(k)$ can be rewritten as

$$\rho(k) = \rho_0(k) - (\rho_0\cdot 2\pi F)(k;-k_F,k_F]. \quad (5.20)$$

Thus, we have

$$E_\Delta - E_0 = L\Delta^2\epsilon'(k_F)\rho(k_F) = (2\pi/L)v_s z(k_F)^2. \quad (5.21)$$

This verifies $v_J = v_s z(k_F)^2$. In view of eq. (4.19), at low energies, the GIG looks like an IEG with

$$\lambda_{eff} = z(k_F)^{-2}. \quad (5.22)$$

It can be shown that it is the effective statistics (5.22) that controls the low-T critical properties of GIG, as λ does for IEG. Linearization near the Fermi points and bosonization of the low-energy effective Hamiltonian go the same way as before for IEG. The only difference now is that the slope of the linearized dispersion for the dressed energy $\epsilon_\pm(k) = \pm\epsilon'(k_F)(k\mp k_F)+\mu = \pm v_s(p(k)\mp p_F)+\mu$, is smooth at $k\sim\pm k_F$. So bosonization is standard and the bosonized Hamiltonian is the same as eq. (4.44) for IEG, only with λ replaced by λ_{eff}. However, before going to the bosonization we need an effective Hamiltonian of the fermions with the dressed energy. Unlike the IEG, in the GIG case, $\epsilon(k,T) = \epsilon(k) + O(T^2/v_s)$. Now, we work out the T-expansion of $\epsilon(k,T)$ explicitly in the low-T limit:

$$\epsilon(k,T) = \epsilon(k) + \tilde\epsilon(k,T) + O(T^3/v_s^2). \quad (5.23)$$

One finds that

$$\tilde\epsilon(k,T) = \frac{\pi T^2}{6\epsilon'(k_F)}f(k), \quad (5.24)$$

with the function f determined by

$$\begin{aligned} f(k) &= \Phi(k_F-k) - (\Phi\cdot f)(k;-k_F,k_F] \\ &= \Phi(k_F-k) - (\Phi\cdot\Phi)(k;-k_F,k_F] \\ &\quad +(\Phi\cdot(\Phi\cdot\Phi)(k;-k_F,k_F] + \ldots \end{aligned} \quad (5.25)$$

Note that the equation that $\rho(k)$ obeys can be rewritten as

$$\frac{\rho(k)}{\rho_0} = 1 - \int_{-k_F}^{k_F} dk'\{\Phi(k-k')$$
$$+(\Phi\cdot\Phi)(k';-k_F,k_F](\Phi\cdot(\Phi\cdot\Phi)(k';-k_F,k_F] + \ldots \quad (5.26)$$

Integrating (5.25) over k and comparing with (5.26), one has

$$\int_{-k_F}^{k_F}\frac{dk}{2\pi}f(k) = 1 - \frac{\rho(k_F)}{\rho_0}, \quad (5.27)$$

and then

$$\int_{-k_F}^{k_F}\frac{dk}{2\pi}\tilde\epsilon(k,T) = \frac{\pi T^2}{6\epsilon'(k_F)}(1-2\pi\rho(k_F)). \quad (5.28)$$

Substituting (5.23) into the thermodynamic potential (4.7), we have

$$\frac{\Omega(T)}{L} = -\frac{T}{2\pi}\int_{-\infty}^{\infty} dk\ln(1+e^{-\epsilon(k)/T}) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{1+e^{\epsilon(k)/T}}\tilde\epsilon(k,T). \quad (5.29)$$

In the low-T limit, the first term in the last equation gives

$$\frac{\Omega(0)}{L} - \frac{\pi T^2}{6\epsilon'(k_F)},$$

with

$$\frac{\Omega(0)}{L} = \frac{1}{2\pi}\int_{-k_F}^{k_F} dk\epsilon(k).$$

and the second term is approximately given by (5.28). Thus,

$$\frac{\Omega(T)}{L} - \frac{\Omega(0)}{L} = -\frac{\pi T^2}{6v_s}, \quad (5.30)$$

which proves the central charge $c=1$ CFT behavior of the theory at the low energy.

We may also confirm this from the finite size scaling in the spatial direction. To see this, we consider the discrete version of the equation in which the density $\rho_L(k_i)$ obeys

$$\rho_L(k_i) = \frac{1}{2\pi} - \sum_{j\neq i}\Phi(k_i-k_j). \quad (5.31)$$

Using the relation between discrete sum and integration

$$\frac{1}{L}\sum_{n=N_1}^{N_2} f(\frac{I_n}{L}) = \int_{(N_1+1/2)/L}^{(N_2-1/2)/L} dx f(x)$$
$$+\frac{1}{24L^2}[f'((N_1-1/2)/L) - f'((N_2+1/2)/L)]$$
$$+O(1/L^3), \quad (5.32)$$

one has

UU IR Author Manuscript

$$\rho_L(k) \approx \frac{1}{2\pi} - (\Phi \cdot \rho_L)(k; -k_F, k_F] - \frac{1}{24L^2}\frac{1}{\rho(k_F)}\left[\frac{\partial\Phi(k-k')}{dk'}\right]_{-k_F} + \frac{1}{24L^2}\frac{1}{\rho(k_F)}\left[\frac{\partial\Phi(k-k')}{dk'}\right]_{k_F}. \quad (5.33)$$

Denote

$$\rho_L = \rho + \rho_1, \quad (5.34)$$

where $\rho(k)$ is of the order $O(1/L^0)$ and $\rho_1(k)$ the order $O(1/L^2)$. Then, $\rho(k)$ is as defined and $\rho_1(k)$ is determined by

$$\rho_1(k) = -\frac{1}{24L^2}\frac{1}{\rho(k_F)}\left[\frac{\partial\Phi(k-k')}{dk'}\right]_{-k_F} + \frac{1}{24L^2}\frac{1}{\rho(k_F)}\left[\frac{\partial\Phi(k-k')}{dk'}\right]_{k_F} - (\Phi\cdot\rho_1)(k; -k_F, k_F], \quad (5.35)$$

The corresponding thermodynamic potential reads

$$\begin{aligned}\frac{\Omega_L(0)}{L} &= \frac{1}{L}\sum_i \epsilon_0(k(\frac{I_i}{L})) \\ &= \int_{-k_F}^{k_F} dk \rho_L(k)\epsilon_0(k) + \frac{1}{24L^2\rho(k_F)}[\epsilon_0'(k)|_{-k_F} - \epsilon_0'(k)|_{k_F}] \\ &= \int_{-k_F}^{k_F} dk\rho(k)\epsilon_0(k) + \int_{-k_F}^{k_F} dk \rho_1(k)\epsilon_0(k) \\ &+ \frac{1}{24L^2\rho(k_F)}[\epsilon_0'(k)|_{-k_F} - \epsilon_0'(k)|_{k_F}]. \quad (5.36)\end{aligned}$$

The first term of the last equation is $\Omega(0)/L$ and the rest, using (5.35), can be written as

$$\begin{aligned}&-\frac{1}{24L^2\rho(k_F)}\frac{\partial}{\partial k}\Big(\epsilon_0(k) + (-1)(\Phi\cdot\epsilon_0) \\ &+(-1)^2((\Phi\cdot\Phi)\cdot\epsilon_0) + ...)(k; -k_F, k_F]\Big)_{k=k_F} \\ &+\frac{1}{24L^2\rho(k_F)}\frac{\partial}{\partial k}\Big(\epsilon_0(k) + (-1)(\Phi\cdot\epsilon_0) \\ &+(-1)^2((\Phi\cdot\Phi)\cdot\epsilon_0) + ...)(k; -k_F, k_F]\Big)_{k=-k_F}. \quad (5.37)\end{aligned}$$

Recall the equation that $\epsilon(k)$ obeys, one has immediately,

$$\frac{\Omega_L(0)}{L} - \frac{\Omega(0)}{L} = -\frac{\pi}{12L^2}\frac{\epsilon'(k)_{k_F} - \epsilon'(k)_{-k_F}}{2\pi\rho(k_F)} = -\frac{\pi v_s}{6L^2}. \quad (5.38)$$

as desired.

Similar to the case of IEG, we also could have a fermion representation of the grand partition function with the temperature-dependent spectrum. To derive the low-energy effective theory, however, one rewrites the thermodynamic potential (5.29) in the low-T limit as

$$\frac{\Omega(T)}{L} \approx \frac{\Omega(0)}{L} - 2T\rho(k_F)I(k_F, T), \quad (5.39)$$

where

$$\begin{aligned}I(k_F, T) &= \int_{k_F-\delta}^{k_F+\delta} dk \ln(1 + e^{-|\epsilon(k)|/T}) \\ &= \int_{p_F-\delta}^{p_F+\delta} \frac{dp}{2\pi}\rho(p)\ln(1 + e^{-|\epsilon(k((p))|/T}). \quad (5.40)\end{aligned}$$

That is,

$$\begin{aligned}\frac{\Omega(T)}{L} &= \int_{-k_F}^{k_F}\frac{dk}{2\pi}\epsilon(k) \\ &- \frac{T}{2\pi}\int_{-p_F+\delta}^{p_F-\delta} dp \ln(1 + e^{-|\epsilon(k(p))|/T}), \quad (5.41)\end{aligned}$$

where p is the physical (dressed) momentum. The grand partition function reads

$$Z_G \simeq \prod_{k'}(1 + e^{-\beta\epsilon(k(k'))}), \quad (5.42)$$

where $k' = k$ for $|k| < k_F - \delta$ and $k' = p$ for $|k| > k_F - \delta$. Now, we can have an effective Hamiltonian because $\epsilon(k)$ is T-independent,

$$H_{\text{eff}} = \sum_{k'} \epsilon(k(k'))c_{k'}^\dagger c_{k'}. \quad (5.43)$$

Similar to the IEG case, the low-T excitations can be considered by taking the linear approximation near the Fermi points, and after bosonization the zero-temperature excitations should be added back. The way to bosonize the linear Hamiltonian is also similar to the case of IEG. Because the dressed energy is smooth at the Fermi points now, the bosonization is even simpler. The density fluctuation operators are simply given by

$$\begin{aligned}\rho_q^{(+)} &= \sum_{k\sim k_F} : c_{p+q}^\dagger c_p : \\ \rho_q^{(-)} &= \sum_{k\sim -k_F} : c_{p-q}^\dagger c_p : . \quad (5.44)\end{aligned}$$

The commutators among $\rho_q^{(\pm)}$ and $H_\pm$ are

$$\begin{aligned}[\rho_q^{(\pm)}, \rho_{q'}^{(\pm)\dagger}] &\approx \langle 0|[\rho_q^{(\pm)}, \rho_{q'}^{(\pm)\dagger}]|0\rangle = \sum_{p_F-q<p<p_F}\langle 0|c_{p+q}c_{p+q'}^\dagger|0\rangle \\ &= \delta_{q,q'}\sum_{p_F-q<p<p_F} 1 = \frac{L}{2\pi}q\delta_{q,q'} \quad (5.45)\end{aligned}$$

and

UU IR Author Manuscript

$$[H_\pm, \rho_q^{(\pm)}] \approx \langle 0|[H_\pm, \rho_q^{(\pm)}]|0\rangle = \pm v_F q \rho_q^{(\pm)}. \quad (5.46)$$

Introducing the normalized bosonic annihilation operators

$$b_q = \sqrt{2\pi/qL}\,\rho_q^{(+)}, \qquad \tilde{b}_q = \sqrt{2\pi/qL}\,\rho_q^{(-)\dagger} \quad (5.47)$$

and adding back the zero-mode contributions, the bosonized Hamiltonian satisfying (5.46) is given by

$$H_B = v_s\{\sum_{q>0} q(b_q^\dagger b_q + \tilde{b}_q^\dagger \tilde{b}_q) + \frac{1}{2}\frac{\pi}{L}[\lambda_{\rm eff} M^2 + \frac{1}{\lambda_{\rm eff}} J^2]\}, \quad (5.48)$$

which agrees with the bosonized Hamiltonian (2.4) in the Luttinger liquid theory. We see that with λ replaced by $\lambda_{\rm eff}$, the bosonized Hamiltonian for the GIG is the same as that for the IEG. So, all consequences we have obtained from the bosonized Hamiltonian in the IEG case can be applied to the GIG case. Especially, there is an (allowed) $\Psi^\dagger_{\lambda_{eff}}$ describing the particle excitation near the Fermi surface with both anyon and exclusion statistics being λ_{eff}. In this sense, one may say that the effect of mutual statistics is to renormalize the statistics matrix.

Here we remark that in IEG, $\Phi(k,k') = (\lambda-1)\,\delta(k-k')$ is not smooth, so the dressed charge has a jump at k_F: $z(k_F^+) = 1$ and $z(k_F^-) = \lambda^{-1}$ for $k_F^\pm = k_F \pm 0^+$. The general Luttinger-liquid relation is of the form

$$v_N = v_s[z(k_F^+)z(k_F^-)]^{-1}, \quad v_J = v_s z(k_F^+)z(k_F^-). \quad (5.49)$$

B. Non-ideal Gas

Finally, we examine non-ideal gases, e.g., with general Luttinger-type density-density interactions,

$$H = H_{\rm eff} + H_I,$$
$$H_I = \frac{\pi}{L}\sum_{q\geq 0}[U_q(\rho_q\rho_q^\dagger + \tilde{\rho}_q\tilde{\rho}_q^\dagger) + V_q(\rho_q\tilde{\rho}_q^\dagger + \tilde{\rho}_q\rho_q^\dagger)], \quad (5.50)$$

where $H_{\rm eff}$ is given by (5.43) describing a GIG, and ρ_q and $\tilde{\rho}_q$ are the excluson density fluctuations near $\pm k_F$ respectively. After bosonization, the total Hamiltonian remains bilinear in densities:

$$\begin{aligned} H &= H_B + H_I \\ &= \frac{1}{2}\sum_{q>0} q[(v_s + U_q)(b_q^\dagger b_q + \tilde{b}_q^\dagger\tilde{b}_q + b_q b_q^\dagger + \tilde{b}_q\tilde{b}_q^\dagger) \\ &\quad + V_q(b_q^\dagger\tilde{b}_q^\dagger + b_q\tilde{b}_q + \tilde{b}_q^\dagger b_q^\dagger + \tilde{b}_q b_q)] + \frac{1}{2}\frac{\pi}{L}[v_N M^2 + v_J J^2] \\ &\quad + \frac{\pi}{L}[U_0(M_R^2 + M_L^2) + 2V_0 M_R M_L] - \sum_{q>0} v_s q \end{aligned} \quad (5.51)$$

Using the Bogoliubov transformation, the Hamiltonian can be easily diagonalized

$$\begin{aligned} H &= \sum_{q>0}\omega_q(a_q^\dagger a_q + \tilde{a}_q^\dagger\tilde{a}_q) + \frac{1}{2}(\pi/L)[\tilde{v}_N M^2 + \tilde{v}_J J^2] + \mathcal{E}_0, \\ \mathcal{E}_0 &= \sum_{q>0}(\omega_q - v_s q), \end{aligned} \quad (5.52)$$

where

$$\begin{aligned} a_q^\dagger &= \cosh\ \tilde{\varphi}_0\ b_q^\dagger - \sinh\ \tilde{\varphi}_0\ \tilde{b}_q^\dagger, \\ \tilde{a}_q^\dagger &= \cosh\ \tilde{\varphi}_0\ \tilde{b}_q^\dagger - \sinh\ \tilde{\varphi}_0\ b_q^\dagger, \end{aligned} \quad (5.53)$$

and the renormalized velocities are

$$\begin{aligned} v_s &\to \tilde{v}_s = |(v_s + U_0)^2 - V_0^2|^{1/2}, \\ v_N &\to \tilde{v}_N = \tilde{v}_s e^{-2\tilde{\varphi}_0}, \\ v_J &\to \tilde{v}_J = \tilde{v}_s e^{2\tilde{\varphi}_0}. \end{aligned} \quad (5.54)$$

with the controlling parameter $\tilde{\varphi}_0$ determined by

$$\tanh(2\tilde{\varphi}_0) = \frac{v_J - v_N - 2V_0}{v_J + v_N + 2U_0}. \quad (5.55)$$

Thus, the Luttinger-liquid relation ((4.19) survives with λ_{eff} of GIG renormalized to

$$\tilde{\lambda}_{eff} = e^{-2\tilde{\varphi}_0}. \quad (5.56)$$

Note that the new fixed point depends both on the position of the Fermi points and on the interaction parameters U_0 and V_0, leading to "non-universal"exponents.

VI. DISCUSSIONS AND CONCLUSIONS

In conclusion, we have shown that 1-d IEG (without mutual statistics) exactly reproduces the low-energy and low-T properties of (one-component) Luttinger liquids. This gives rise to the following physical picture: At low temperature, the Luttinger liquids can be approximately thought of as an IEG consisting of quasiparticle excitations. Introducing mutual statistics or/ and Luttinger-type interactions among these excitations only shifts the value of λ_{eff}. Thus the essence of Luttinger liquids is to have an IEG obeying FES as their fixed point. This is our characterization of Luttinger liquids in terms of FES.

In this way, we have explicitly answered the three questions raised in the introduction about Luttinger liquids:

- The physical meaning of the Haldane's controlling parameter is the quasiparticle's effective statistics, λ_{eff}.
- The Luttinger liquids, more precisely, the IEG, indeed describe the infrared (or low-energy) fixed

points in 1-d systems, since their effective field theory at low energy is conformally invariant. However, these fixed points are *not* isolated; they form a fixed-point line. Both the chemical potential and coupling constants are relevant perturbations that can drive the fixed point to move along the line, corresponding to the "renormalization" of the effective statistics λ_{eff} and leading to "non-universal" exponents.

- It is conceivable that some strongly correlated systems, exhibiting non-Fermi liquid behavior, in two or higher dimensions may also be characterized as having a GIG with appropriate statistics matrix as their low-energy or low-temperature fixed point. This is because the concept of exclusion statistics is independent of spatial dimeniosnality of the system.

Moreover, we also showed that the effective field theory of 1-d IEG is a CFT with central charge $c = 1$ and compactified radius $R = \sqrt{1/\lambda}$. The particle-hole duality of the exclusons implies the CFT has an unusual duality $R \leftrightarrow 1/R$, meaning that the CFT belongs to a new variant of the $c = 1$ CFT's, i.e the ones that are compactified on an "orbifold" S^1/Z_2 rather than on a circle. Physically, the differences are due to different constraints on the zero-mode quantum numbers. The CFT explanation makes a better understanding of the single-particle operators, especially, the anyonic (or exclusonic) ones. Also, the CFT techniques provide a systematic way to calculate the correlation functions.

Finally we observe several additional implications of this work: 1) Our bosonization and operator derivation of CFT at low energies or in low-T limit can be applied to Bethe ansatz solvable models, including the long-range (e.g., Calogero-Sutherland) one [9]. 2) Here we have only consider one-species cases, i.e., with excitations having no internal quantum numbers such as spin. Our bosonization and characterization of Luttinger liquids are generalizable to GIG with multi-species, with the effective statistics matrix related to the dressed charge matrix [9]. 3) The chiral current algebra in eqs. (4.41) and (4.42)with $\lambda = 1/m$ coincides with that derived by Wen [33] for edge states in $\nu = 1/m$ fractional quantum Hall fluids. So these edge states and their chiral Luttinger-liquid fixed points can be described in terms of chiral IEG.

This work was supported in part by the U.S. NSF grant PHY-9309458, PHY-9970701 and NSF of China.

APPENDIX A: THE HARMONIC FLUID DESCRIPTION

In coordinate space, there is a harmonic fluid description [16] of the Luttinger liquid. Instead of the θ-ϕ representation that Haldane originally used, we prefer the right-left-moving representation. The density operator can be written as the Fourier transformations

$$\begin{aligned}
\rho(x) &= \rho_R(x) + \rho_L(x), \\
\rho_R(x) &= \frac{M_R}{L} + \sum_{q>0} \sqrt{\frac{q}{2\pi L e^{-2\varphi}}} (e^{iqx} b_q + e^{-iqx} b_q^\dagger), \\
\rho_L(x) &= \frac{M_L}{L} + \sum_{q>0} \sqrt{\frac{q}{2\pi L e^{-2\varphi}}} (e^{-iqx} \tilde{b}_q + e^{iqx} \tilde{b}_q^\dagger), \qquad \text{(A1)}
\end{aligned}$$

where $M_{R,L}$ are given by $M = M_R + M_L$ and $\tilde{b}_q = b_{-q}$ for $q > 0$.

The boson field $\phi(x)$, which is conjugated to $\rho(x)$ and satisfies

$$[\phi(x), \rho(x')] = i\delta(x - x'), \qquad \text{(A2)}$$

is given by

$$\begin{aligned}
\phi(x) &= \phi_R(x) + \phi_L(x), \\
\phi_R(x) &= \frac{\phi_0}{2} + \frac{\pi J_R x}{L} + i \sum_{q>0} \sqrt{\frac{\pi e^{-2\varphi}}{2qL}} (e^{iqx} b_q - e^{-iqx} b_q^\dagger), \\
\phi_L(x) &= \frac{\phi_0}{2} + \frac{\pi J_L x}{L} + i \sum_{q>0} \sqrt{\frac{\pi e^{-2\varphi}}{2qL}} (e^{-iqx} \tilde{b}_q - e^{iqx} \tilde{b}_q^\dagger),
\end{aligned} \qquad \text{(A3)}$$

with $J = J_R + J_L$. We have to assign the quantum numbers such that there are only two independent variables in $M_{R,L}$ and $J_{R,L}$. A consistent choice is

$$M_R = J_R, \qquad M_L = -J_L. \qquad \text{(A4)}$$

Then,

$$J = J_R + J_L, \qquad M = J_R - J_L. \qquad \text{(A5)}$$

Here ϕ_0 is an angular variable conjugated to M: $[\phi_0, M] = i$. The Hamiltonian (2.4) becomes

$$H = \frac{1}{2} \int_0^L dx \, [\pi v_N \rho(x)^2 + v_J/\pi \, (\partial_x \phi(x))^2], \qquad \text{(A6)}$$

or by a field rescaling,

$$H = \frac{v_s}{2\pi} \int_0^L dx \, [\Pi(x)^2 + (\partial_x X(x))^2], \qquad \text{(A7)}$$

where

$$\Pi(x) = \pi e^{-\varphi} \rho(x), \qquad X(x) = e^{\varphi} \phi(x). \qquad \text{(A8)}$$

With $X(x,t) = e^{iHt} X(x) e^{-iHt}$, the Lagrangian density reads

$$\mathcal{L} = \frac{v_s}{2\pi} \, \partial_\alpha X(x,t) \, \partial^\alpha X(x,t), \qquad \text{(A9)}$$

which describes a free scalar field theory in 1 + 1-dimensions.

UU IR Author Manuscript

[1] F. D. M. Haldane, J. Phys. **C 14**, 2585 (1981).
[2] F. D. M. Haldane, Phys. Rev. Lett. **67**, 937 (1991); and in Proc. 16th Taniguchi Symposium, eds. N. Kawakami and A. Okiji, Springer Verlag (1994).
[3] Y. S. Wu, Phys. Rev. Lett. **73**, 922 (1994).
[4] C. N. Yang and C. P. Yang, J. Math. Phys. **10**, 1115 (1969).
[5] D. Bernard and Y. S. Wu, in Proc. 6th Nankai Workshop, eds. M. L. Ge and Y. S. Wu, World Scientific (1995).
[6] C. Nayak and F. Wilczek, Phys. Rev. Lett. **73**, 2740 (1994).
[7] Z. N. C. Ha, Nucl. Phys. B**435**, 604 (1995).
[8] Y. Hatsugai, M. Kohmoto, T. Koma and Y. S. Wu, Phys. Rev. B **54**, 5358 (1996).
[9] Y. Yu , H. X. Yang and Y. S. Wu, cond-mat/9911141.
[10] A. Dasnierea de Veigy and S. Ouvry, Phys. Rev. Lett. **72**,600(1994).
[11] Y. S. Wu, Y. Yu, Y. Hatsugai and M. Kohmoto, Phys. Rev. B **57**, 9907 (1998).
[12] Y. S. Wu and Y. Yu, Phys. Rev. Lett. **75**, 890 (1995).
[13] See, e.g. , R. Shankar, Rev. Mod. Phys. **66**, 129, (1994).
[14] S. Tomonaga, Prog. Theor. Phys. **5**, 544 (1950).
[15] D. C. Mattis and E. H. Lieb, J. Math. Phys. **6**, 304 (1965).
[16] F. D. M. Haldane, Phys. Rev. Lett. **47**, 1840 (1981).
[17] F. Calogero, J. Math. Phys. **10** 2197 (1967).
[18] B. Sutherland, J. Math. Phys. **12**, 246, 251 (1971).
[19] See, e.g. , R. Dijkgraaf, E. Verlinde and H. Verlinde, Commun. Math. Phys. **115**, 649 (1988). The standard duality in usual $c = 1$ CFT is $R \leftrightarrow 2/R$.
[20] N. Kawakami and S. K. Yang, Phys. Rev. Lett. **67**, 2493 (1990).
[21] R. Shankar and M.V.N.Murthy, Phys. Rev. Lett. **72**, 3629 (1994).
[22] S. Iso, Nucl. Phys. B **443**, 581 (1995).
[23] S. Iso and S. J. Rey, Phys. Lett. B **352**,111 (1995).
[24] R. Caracciolo, A. Lerda and G. R. Zemba, Phys. Lett. B **352**, 304 (1995).
[25] See, e.g., H. J. Schulz, in "Proceedings of Les Houches Summer School LXI", ed. E. Akkermans, G. Montambaux, J. Pichard, and J. Zinn-Justin (Elsevier, Amsterdam, 1995), p.533.
[26] J. M. Luttinger, J. Math. Phys. **4**, 1154 (1963).
[27] Y. S. Wu, Invited Lectures in "Topics in Theoretical Physics", (Proceedings of the Second Pacific Winter School for Theoretical Physics; Sorak Mountains, Korea; Jan. 19-24, 1995), ed. by Y. M. Cho (World Scientific, 1996); pp. 27-59.
[28] It is easy to understand why it is δF_0 rather than δE_0 that exactly contains the zero-mode contributions in (2.4): In writing down the Hamiltonian (2.4), we have shifted the zero point of the energy to the Fermi surface.
[29] Alternatively, we may extend the Fermi sea to that for two separate right- and left-moving fermions with linear dispersion for k covering the whole real axis in each sector. Then our commutation relations are exact. (See [15])
[30] Here operators O_m give rise to the Klein factors necessary for correct commutation relation for Ψ_F and $\Psi_F^\dagger$.
[31] N. Read, Phys. Rev. Lett., **62**, 86 (1988).
[32] F. D. M. Haldane, Phys. Lett. **81 A**, 153 (1981); N. M. Bogoliubov, A. G. Izergin and V. E. Korepin, J. Phys. **A 20**, 5361 (1987).
[33] X.G. Wen, Phys. Rev. Lett. **64** , 2206 (1990).

UU IR Author Manuscript